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A Linear Programming Method for Finding Orthocomplements in Finite Lattices

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A method of embedding partially ordered sets into linear spaces is presented. The problem of finding all orthocomplementations in a finite lattice is reduced to a linear programming problem.

1. INTRODUCTION AND BASIC DEFINITIONS

We introduce linear algebraic tools for finite lattices. The idea of the proposed method looks as follows. Given a finite latice L, we consider its linear hull H (L) as the collection of all real-valued functions on L. Any mapping f:L $\to L$ is associated with a linear operator f:H$(L) \to H(L)$. The collection $\mathcal{H}$ of all linear operators in $H(L)$ is a linear space itself, and we introduce a convex set of precomplements as a subset of $\mathcal{H}$. Then it turns out that the complements in the lattice L are in 1–1 correspondence with the solutions of a linear programming problem in the space $\mathcal{H}$.

Now we introduce the basic definitions. Let $H = H(L)$ be the set of all real functions on L. Fix up a preferred basis in H labeled by the elements of L: for any $p \in L$ its counterpart is the delta function δp:

$$\delta_p(q) = \begin{cases} 1 & \text{if} \quad p = q \\ 0 & \text{otherwise} \end{cases}$$

The space H possesses the natural structure of commutative algebra since functions can be pointwise multiplied: $(f \cdot g)(p) = f(p) \cdot g(p)$ for any $f, g \in H$. The unit element $\mathbf{1}$ of H is $\mathbf{1} = \Sigma_{p \in L}\, \delta_p$.

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The partial order $\leq$ in L is associated with the zeta operator (Aigner, 1979) in H, whose matrix is the incidence matrix of the partial order of L:

$$\zeta(p, q) = \begin{cases} 1 & \text{if} \quad p \leq q \\ 0 & \text{otherwise} \end{cases}$$

The operator ζ has the following properties: its matrix is upper triangle (under an appropriate enumeration of the elements of L), and has 1 on the diagonal. Therefore ζ is always invertible. Its inverse is denoted by $\mu = \zeta^{-1}$. The matrix of μ as a function of two variables ranging over L is called the *Möbius function* of L (Stanley, 1986).

1. THE POLYTOPE OF PRECOMPLEMENTS

Two elements $p, q \in L$ are said to be *disjoint* if $p \wedge q = 0$, *conjoint* if $p \vee q = 1$ (where 0, 1 are the least and the greatest elements of L, resp.) and *complemented* if they are both disjoint and conjoint. A *complement* on the lattice L is an idempotent permutation α of the elements of L such that (i) $p \leq q$ implies $\alpha q \leq \alpha p$ and (ii) any pair $p, \alpha p$ is complemented (it suffices to require them be disjoint). With any permutation α on L we can associate a linear operator α: $H \to H$ defined on the basis of delta functions as follows: $\alpha(\delta_p) := \delta_{\alpha p}$. When α is a complement on L, the matrix of α is an idempotent orthogonal matrix, therefore satisfying the following conditions: $\alpha \geq 0$ (since the elements of α equal to 0 or 1), $\alpha \mathbf{1} = \mathbf{1}$ (since there is only one 1 at each row), and $\alpha^\top = \alpha$. The next necessary condition for α to be a complement is that it reverses order. In operator form it is expressed as follows: $\alpha\zeta = \zeta^\top\alpha$.

Furthermore, since α is a complement, for any $p \in L$ we have $q \leq p$, $q \leq \alpha p \Rightarrow q = 0$, which means $\zeta\alpha\delta_p \cdot \zeta\delta_p = \delta_0$, hence

$$\operatorname{tr}(\zeta^\top\zeta\alpha) = \sum_p (\zeta\alpha\delta_p, \zeta\delta_p) = \sum_p (\zeta\alpha\delta_p \cdot \zeta\delta_p, \mathbf{1}) = \sum_p (\delta_0, 1) = n$$

where p ranges over L, and n is the cardinality of L.

Now we can introduce the convex subset $\mathcal{P}$ of $\mathcal{H} = \mathrm{Mat}_n$ defined as follows: $\alpha \in \mathcal{P}$ iff

$$\begin{cases} \alpha \geq 0 & \alpha\mathbf{1} = \mathbf{1} \\ \alpha = \alpha^\top & \alpha\zeta = \zeta^\top\alpha \\ \operatorname{tr}(\zeta^\top\zeta\alpha) = n \end{cases} \tag{1}$$

Evidently, all complements are in $\mathcal{P}$; however, $\mathcal{P}$ is a continuous subset of Mat_n, namely, a polytope, since all the equations (1) are linear. We call the polytope $\mathcal{P}$ the *polytope of precomplements.*

2. MAIN RESULTS

These are two theorems demonstrate the power of the linear approach in the theory of posets.

Theorem 1. The orthocomplements, and only they, are the integer vertices of the polytope $\mathcal{P}$.

The Idea of the Proof. Otherwise the condition $\alpha\mathbf{1} = \mathbf{1}$ will be broken.

Theorem 2. The orthocomplements of L are the optimal solutions of the following linear programming problem:

$$\mathrm{tr}(\zeta\zeta^{\top}\alpha) \to \min$$

$$\begin{cases} \alpha\mathbf{1} = \mathbf{1} \\ \alpha^{\top} = \alpha \\ \alpha\zeta = \zeta^{\top}\alpha \end{cases}$$

Sketch of the Proof. The condition $\alpha\mathbf{1} = \mathbf{1}$ implies tr $(\zeta\zeta^{\top}\alpha) \geq n$, and then recall the previous theorem.

3. CONCLUDING REMARKS

The proposed techniques of linear embedding of finite lattices can be extended to posets. All the results still will be valid, but we face a strange effect: in the linear hull of *any* poset the operations $\wedge$ and $\vee$ are always defined, but the results of these operations may bring us beyond the poset in question and yield the weighted sums of the elements of the poset. This issue needs further investigation.

REFERENCES

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